Published by Institute of Physics Publishing for SISSA

Received: *August 15, 2005*
Accepted: *September 12, 2005*
Published: *October 4, 2005*

Surface charges and dynamical Killing tensors for higher spin gauge fields in constant curvature spaces

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Abstract: In the context of massless higher spin gauge fields in constant curvature spaces, we compute the surface charges which generalize the electric charge for spin one, the color charges in Yang-Mills theories and the energy-momentum and angular momentum for asymptotically flat gravitational fields. We show that there is a one-to-one map from surface charges onto divergence free Killing tensors. These Killing tensors are computed by relating them to a cohomology group of the first quantized BRST model underlying the Fronsdal action.

Keywords: Field Theories in Higher Dimensions, Space-Time Symmetries, Gauge Symmetry.

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http://jhep.sissa.it/archive/papers/jhep102005010/jhep102005010.pdf

JHEP10(2005)010

Contents

1. Introduction **1**

2. Higher spin gauge fields in constant curvature spaces **3**

3. From dynamical to standard Killing tensors **5**

4. Killing tensors in constant curvature spaces **6**

4.1 Flat space 6

4.2 Curved space 7

5. Surface charges for higher spin gauge fields **10**

6. Conclusion and perspectives **11**

A. Cauchy order of higher spin gauge fields **12**

B. Killing tensors in flat space **14**

1. Introduction

In irreducible gauge theories, there are two kinds of charges. The standard Noether charges for global symmetries are given by the integral of the time component of a conserved current j^μ, $\partial_\mu j^\mu \approx 0$. Here, ∂_μ denotes the total derivative and ≈ 0 means zero "on-shell", i.e., on all solutions of the field equations. More invariantly, in n spacetime dimensions, these conserved charges correspond to the integral over an $n-1$ dimensional hypersurface of the $n-1$ form $j^{n-1} = \sqrt{|g|}\frac{1}{(n-1)!}\epsilon_{\mu\mu_2\dots\mu_n} j^\mu dx^{\mu_2}\dots dx^{\mu_n}$. The properties of the charges then follow from Stokes theorem and on-shell closure of j^{n-1}, $dj^{n-1} \approx 0$.

The second kind of charges are given by the integral over an $n-2$ dimensional hypersurface of an on-shell closed $n-2$ form k^{n-2}, $dk^{n-2} \approx 0$. These charges are associated with gauge symmetries, or more precisely, with "reducibility parameters", i.e., field dependent parameters of gauge symmetries restricted by the condition that the associated gauge symmetries leave the fields invariant on-shell. For electromagnetism, a constant gauge parameter has this property, while for linearized gravity, examples of such parameters which are field independent are provided by the Killing vectors of the background metric.

It is important to realize that the surface charges of the linearized theories are also relevant in non-abelian Yang-Mills theories or in full gravity. Indeed, they can be used at infinity for configurations that asymptotically approach some background so that the linearized theory applies. A clear early discussion for massless spin 1 and spin 2 gauge fields has been given by Abbott and Deser [1, 2]. These charges are also useful in order to study physical properties of exact solutions to the full interacting theory [3–5].

A more systematic investigation of surface charges involves the question when on-shell closed forms k^{n-2} should be considered trivial. The natural answer is if they are on-shell exact, $k^{n-2} \approx dl^{n-3}$. Surface charges are thus identified with elements of the so-called characteristic cohomology group in form degree $n-2$, $H^{n-2}_{\text{weak}}(d)$, where "weak" means that equalities are required to hold merely on-shell. Characteristic cohomology has been discussed extensively in the mathematical literature [6 − 10].

In regular Lagrangian gauge field theories, it has been shown [11 − 13] that there is a one-to-one map from $H^{n-2}_{\text{weak}}(d)$ onto equivalence classes of reducibility parameters. Reducibility parameters should be considered equivalent when they coincide on-shell. Under some technical assumptions, it has then been shown that irreducible gauge theory do not admit characteristic cohomology in form degree lower than $n-2$. It has also been shown explicitly in flat [14] and in constant curvature spaces [15] of dimensions greater or equal to 3 that the non trivial reducibility parameters of massless spin 2 fields can without loss of generality be taken not to depend on the fields and thus reduce to the standard Killing vectors of the background metric. Since a similar result can easily be established for the spin 1 case, it then follows that for spin 1 and 2, the surface charges are exhausted by those found by Abbott and Deser.

The aim of this paper is to generalize these results to the case of massless higher spin gauge theories in constant curvature spaces in the Fronsdal formulation [16, 17]. As explained above, the surface charges for the free theory will also be relevant for the full interacting theory and exact solutions thereof, once a complete Lagrangian formulation will be found (see the recent work in anti-De Sitter spaces by Vasiliev [18 − 20] for progress in this direction).

In the first section 1, we briefly review the string inspired BRST first quantized description of massless higher spin gauge fields both in flat [21 − 23] and in (anti-) De Sitter spacetimes [24 − 27], with or without trace constraints.

We then show in appendix A that the Cauchy order of these theories is 2 for spin $\geqslant 1$. According to general theorems [11, 13], this implies in particular that there is no characteristic cohomology in degree strictly lower than $n-2$.

In the section 2, the reducibility parameters for higher spin gauge fields are related to dynamical Killing tensors that are divergence free on-shell. The result of the appendix is then used to show that the non trivial reducibility parameters of higher spin gauge theories can be chosen independent of the fields in space-time dimensions greater or equal to 3. Hence, the classification of surface charges for higher spin fields does not depend on the dynamics and is exhausted by standard divergence free Killing tensors of the constant curvature background.

The Killing tensors are explicitly constructed in section 3. In flat space, we use an investigation [28] of the first quantized model underlying the Fronsdal higher spin theories, where the Killing tensor equations have been explicitly solved during the computation of the cohomology group $H^{-3/2}(\Omega)$. For completeness, we re-derive these results in a form suitable for the present context in appendix B. We then apply the standard procedure consisting of inducing the results for constant curvature spaces from the results of flat space in one dimension more.

JHEP10(2005)010

Both in anti-De Sitter [29, 30] (see also [31]) and in flat backgrounds [32], the divergence free Killing tensors constructed in this way are parametrized by the same Young tableaux as the "global higher spin symmetries" discussed originally in the context of the formulation of higher spin gauge theories in terms of frame-like gauge fields and generalized connections. This is no coincidence because global higher spin symmetries leave invariant these fields, and thus also Fronsdal's fields, which emerge as particular combinations thereof.[1]

Finally, in the last section 4, we compute the explicit expressions of the surface charges using general formulas [33] valid for generic irreducible gauge theories.

2. Higher spin gauge fields in constant curvature spaces

The action for massless, double traceless fields $\varphi_{\mu_1\ldots\mu_s}$ of "spin" s in constant curvature spaces has been derived by Fronsdal [17] in 4 dimensions. Its generalization in arbitrary dimensions has been originally constructed in [29]. It is invariant under the gauge transformations

$$\delta_\Lambda \varphi_{\mu_1\ldots\mu_s} = s\nabla_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)}\,, \tag{2.1}$$

with $\Lambda_{\mu_2\ldots\mu_s}$ a traceless gauge parameter. In analogy with string field theory, the Fronsdal gauge field theory can be compactly reformulated as the field theory associated to a BRST first quantized relativistic particle with internal degrees of freedom. In the case of non-vanishing constant curvature, we mostly follow here the notations and conventions of Sagnotti and Tsulaia [27, section 3] to which we refer for further details.

The first quantized system consists of the conjugate variables

$$[x^\mu, p_\nu] = \imath\delta^\mu_\nu\,, \qquad [\alpha^\mu_1, \alpha^\mu_{-1}] = g^{\mu\nu}\ (\text{bosonic})\,, \qquad [c_k, b_l] = \delta_{kl}\ (\text{fermionic})\,, \tag{2.2}$$

where $\mu = 0, \cdots, n-1$, $k, l = -1, 0, 1$ and $g^{\mu\nu}$ is the inverse of the constant curvature metric $g_{\mu\nu}$ used to raise and lower indexes.

The representation space for these commutation relations is chosen to be the space of x dependent wave functions taking values in the Fock space for the oscillators. This Fock space is defined by $b_0|0\rangle = 0 = c_1|0\rangle = b_1|0\rangle = \alpha^\mu_1|0\rangle$. The space-time inner product involves the generally covariant volume element $\sqrt{|g|}d^n x$, while the Fock space inner product is defined by $\langle 0, c_0 0\rangle = 1$. The coordinates x^μ act on the states by multiplication and the action of p_μ on states is defined according to

$$p_\mu = -\imath\left(\frac{\partial}{\partial x^\mu} - \Gamma^\rho{}_{\mu\nu}\alpha^\nu_{-1}\alpha_{1\rho}\right). \tag{2.3}$$

The hermitean nilpotent BRST charge of the system is given by

$$\begin{aligned}\Omega = c_0\left(\widetilde{l}_0 - \frac{4}{L^2}N + \frac{6}{L^2}\right) + c_1 l_{-1} + c_{-1} l_1 - c_{-1}c_1 b_0 - \\ - \frac{6}{L^2}c_0 c_{-1} b_1 - \frac{6}{L^2}c_0 b_{-1} c_1 + \frac{4}{L^2}c_0 c_{-1} b_1 N + \frac{4}{L^2}c_0 b_{-1} c_1 N - \\ - \frac{8}{L^2}c_0 c_{-1} b_{-1} M + \frac{8}{L^2}c_0 c_1 b_1 M^\dagger + \frac{12}{L^2}c_0 c_{-1} b_{-1} c_1 b_1\,,\end{aligned} \tag{2.4}$$

[1]The authors are grateful to M. Vasiliev for pointing this out.

where

$$l_{\pm 1} = \alpha_{\pm 1} \cdot p, \quad [l_1, l_{-1}] = \widetilde{l}_0\,, \qquad N = \alpha_{-1}\alpha_1 + \frac{n}{2}\,, \qquad M = \frac{1}{2}\alpha_1\alpha_1\,, \tag{2.5}$$

and L is the radius of anti-De Sitter space, $\eta_{\alpha\beta} z^\alpha z^\beta \equiv z^2 = -L^2$, with $\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, \ldots, +1, -1)$. The results for de Sitter space are then obtained by replacing L^2 by $-L^2$ in the expression for the BRST charge, while the flat space results are recovered by putting $\frac{1}{L^2} = 0$.

Besides the BRST charge, there are three other operators relevant for our discussion: the anti-hermitian ghost number operator defined by

$$\mathcal{G} = c_0 b_0 + c_1 b_{-1} + c_{-1} b_1 - \frac{1}{2}\,, \tag{2.6}$$

with $[\mathcal{G}, \Omega] = \Omega$; the hermitian total occupation number operator,

$$\mathcal{N}_s = \alpha_{-1} \cdot \alpha_1 + c_{-1} b_1 + b_{-1} c_1 - s\,, \tag{2.7}$$

with $[\mathcal{N}_s, \Omega] = 0 = [\mathcal{N}_s, \mathcal{G}]$ and finally, the trace operator

$$\mathcal{T} = \frac{1}{2}\alpha_1\alpha_1 + b_1 c_1\,, \tag{2.8}$$

with $[\Omega, \mathcal{T}] = 0 = [\mathcal{G}, \mathcal{T}]$, $[\mathcal{N}_s, \mathcal{T}] = -2\mathcal{T}$.

Let us associate to the most general ghost number $-1/2$ state

$$\begin{aligned} |\psi_{-1/2}\rangle = \int d^n x \Big(& \frac{1}{s!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_s} \psi_{\mu_1 \ldots \mu_s}(x) + \\ & + \frac{-\imath}{(s-1)!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-1}} c_0 b_{-1} c_{\mu_1 \ldots \mu_{s-1}}(x) \ + \\ & + \frac{1}{(s-2)!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-2}} c_{-1} b_{-1} d_{\mu_1 \ldots \mu_{s-2}}(x) \Big) |x\rangle |0\rangle \end{aligned} \tag{2.9}$$

of the Hilbert space the string field

$$\begin{aligned} |\Psi_{-1/2}\rangle = \int d^n x \ \sum_s \Big(& \frac{1}{s!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_s} \varphi_{\mu_1 \ldots \mu_s}(x) \ + \\ & + \frac{-\imath}{(s-1)!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-1}} c_0 b_{-1} C_{\mu_1 \ldots \mu_{s-1}}(x) \ + \\ & + \frac{1}{(s-2)!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-2}} c_{-1} b_{-1} D_{\mu_1 \ldots \mu_{s-2}}(x) \Big) |x\rangle |0\rangle\,, \end{aligned} \tag{2.10}$$

with φ, C, D even bosonic and real tensor fields. If one now imposes the constraints $\mathcal{T}|\Psi^T_{-1/2,s}\rangle = 0 = \mathcal{N}_s|\Psi^T_{-1/2,s}\rangle$, $\mathcal{G}|\Psi^T_{-1/2,s}\rangle = -\frac{1}{2}|\Psi^T_{-1/2,s}\rangle$, the Fronsdal action for massless fields of spin s in a constant curvature space can be compactly written with the help of auxiliary fields as

$$S[\varphi, C, D] = -\frac{1}{2}\langle \Psi^T_{-1/2,s}, \Omega \Psi^T_{-1/2,s}\rangle\,. \tag{2.11}$$

If

$$|\lambda_{-3/2}\rangle = \int d^n x \sum_s \left(\frac{\imath}{(s-1)!}\alpha_{-1}^{\mu_1} \ldots \alpha_{-1}^{\mu_{s-1}} b_{-1} \lambda_{\mu_1 \ldots \mu_{s-1}}(x) \right) |x\rangle |0\rangle\ , \tag{2.12}$$

is the general ghost number $-\frac{3}{2}$ state and

$$|\Lambda_{-3/2}\rangle = \int d^n x \sum_s \left(\frac{\imath}{(s-1)!} \alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_{s-1}} b_{-1} \Lambda_{\mu_1 \dots \mu_{s-1}}(x) \right) |x\rangle |0\rangle \,, \tag{2.13}$$

an associated string field, with Λ even bosonic and real gauge parameters, the gauge transformations for this action can be written as

$$\delta_\Lambda \left| \Psi^T_{-1/2,s} \right\rangle = \Omega \left| \Lambda^T_{-3/2,s} \right\rangle \,, \tag{2.14}$$

where $\mathcal{T}|\Lambda^T_{-3/2,s}\rangle = 0 = \mathcal{N}_s |\Lambda^T_{-3/2,s}\rangle$, $\mathcal{G}|\Lambda^T_{-3/2,s}\rangle = -\frac{3}{2}|\Psi^T_{-1/2,s}\rangle$.

Removing the trace constraints on the string fields and the gauge parameters, the action $S = -\frac{1}{2}\langle \Psi_{-1/2,s}, \Omega \Psi_{-1/2,s}\rangle$ continues to describe a perfectly consistent gauge model in a constant curvature space, with gauge transformations given by $\delta_\Lambda |\Psi_{-1/2,s}\rangle = \Omega|\Lambda_{-3/2,s}\rangle$, even though the model contains some reducibility from the point of view of representations for different values of s. More precisely, in 4 dimensional flat space for instance, the model without trace constraints describes massless particles of helicities $-s, -s+2, \dots, s-2, s$. The trace constraints project out the intermediate helicity states (see e.g. [23, 34, 35] and references therein).[2]

Removing the occupation number constraint, the action $S = -\frac{1}{2}\langle \Psi^{(T)}_{-1/2}, \Omega \Psi^{(T)}_{-1/2}\rangle$ describes the sum of all these gauge models. In the 4 dimensional flat case with trace constraints, it describes massless particles of all helicities $\pm s$ precisely once, while in the 4 dimensional anti-de Sitter case, it describes singletons [17].

Finally, removing the ghost number constraint and associating a string field with all the states of the Hilbert space, the individual fields being chosen real, of ghost number $-1/2$ *minus the ghost number of the corresponding state* and of parity this ghost number modulo 2, $S = -\frac{1}{2}\langle \Psi^{(T)}_{(s)}, \Omega \Psi^{(T)}_{(s)}\rangle$ is the Batalin-Vilkovisky master action [46 − 49] (see also [50, 51]) for the gauge models with or without trace constraints.

3. From dynamical to standard Killing tensors

In terms of the fields, the gauge transformations read explicitly

$$\begin{aligned} \delta_\Lambda \varphi_{\mu_1 \dots \mu_s} &= s \nabla_{(\mu_1} \Lambda_{\mu_2 \dots \mu_s)} \,, \\ \delta_\Lambda D_{\mu_1 \dots \mu_{s-2}} &= \nabla^\mu \Lambda_{\mu \mu_1 \dots \mu_{s-2}} \,, \\ \delta_\Lambda C_{\mu_1 \dots \mu_{s-1}} &= \left[\Box + \frac{(s-1)(3-s-n)}{L^2} \right] \Lambda_{\mu_1 \dots \mu_{s-1}} + \frac{(s-1)(s-2)}{L^2} g_{(\mu_1 \mu_2} \Lambda'_{\mu_3 \dots \mu_{s-1})} \,. \end{aligned} \tag{3.1}$$

where $\Lambda'_{\mu_3 \dots \mu_{s-1}} = g^{\mu_1 \mu_2} \Lambda_{\mu_1 \mu_2 \dots \mu_{s-1}}$. Reducibility parameters are defined by gauge parameters $\Lambda_{\mu_1 \dots \mu_{s-1}} = \Lambda_{\mu_1 \dots \mu_{s-1}}[x, \varphi, C, D]$ that may depend on the fields and their derivatives

[2] How to describe massless particles of helicities $\pm s$ without imposing trace constraints is treated in [36 − 38, 34, 39]. Other aspects of higher spin fields in (A)dS spaces are discussed for instance in [40 − 45].

and that satisfy

$$
\begin{aligned}
&\nabla_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)} \approx 0\,,\\
&\nabla^{\mu}\Lambda_{\mu\mu_1\ldots\mu_{s-2}} \approx 0\,,\\
&\left[\Box + \frac{(s-1)(3-s-n)}{L^2}\right]\Lambda_{\mu_1\ldots\mu_{s-1}} + \frac{(s-1)(s-2)}{L^2}g_{(\mu_1\mu_2}\Lambda'_{\mu_3\ldots\mu_{s-1})} \approx 0\,.
\end{aligned}
\tag{3.2}
$$

In appendix A, we show that the fields φ, C, D and their derivatives can be split into two groups (y_A, z_a) so that the y_A can be taken as coordinates on the surface defined by the field equations and the z_a as coordinates off this surface. Because $H^{n-2}_{\text{weak}}(d)$ is isomorphic to reducibility parameters, up to weakly vanishing ones, non trivial reducibility parameters can be chosen independent of the z_a, $\Lambda_{\mu_1\ldots\mu_{s-1}} = \Lambda_{\mu_1\ldots\mu_{s-1}}(x, y)$.

We also show in appendix A that higher spin gauge theories are of Cauchy order 2, $\partial_\mu y_A \subset y_A$, for $\mu \geqslant 2$. In spacetime dimensions greater or equal to 3, the first equation of (3.2) then implies that the reducibility parameters cannot depend y_A either and are thus functions of x^μ alone, $\Lambda_{\mu_1\ldots\mu_{s-1}} = \Lambda_{\mu_1\ldots\mu_{s-1}}(x)$. As a consequence, the weak equalities can be replaced by strong ones in equation (3.2). Furthermore, according to general results [11, 13], theories of Cauchy order 2 have trivial characteristic cohomology in form degrees strictly less than $n-2$.

4. Killing tensors in constant curvature spaces

4.1 Flat space

In flat space with $n \geqslant 3$, we have to solve the equations

$$
\partial_{(\mu_1}\Lambda_{\mu_2\ldots\mu_s)} = 0\,, \qquad \partial^{\mu}\Lambda_{\mu\mu_2\ldots\mu_{s-1}} = 0\,, \qquad \Box\Lambda_{\mu_1\mu_2\ldots\mu_{s-1}} = 0\,,
\tag{4.1}
$$

where the symmetric gauge parameters $\Lambda_{\mu_1\mu_2\ldots\mu_{s-1}}(x)$ are assumed to be traceless or not, depending on whether the model with or without trace constraints is considered.

These equations have been explicitly solved as a second step in the computation of $H^{-3/2}(\Omega)$ in section 4.4 of reference [28].[3] This is no coincidence, but it is related to the way the Batalin-Vilkovisky master action for higher spin gauge fields is constructed out of the first quantized BRST charge on the one hand and the relation between the characteristic cohomology and the local BRST cohomology of the anti-field formalism [11, 13] on the other hand. General results on relating field theory BRST cohomology groups to those of the first quantized model will be given elsewhere. In order to be self-contained, we will explicitly solve these equations again in appendix B. In fact, we will solve them under slightly more general assumptions since we allow for Λ to depend smoothly on x^μ, contrary to reference [28] where formal power series were considered.

[3] The ghost numbers of the first quantized model differ because, in reference [28], the non-Lagrangian context was emphasized and there was no need to have a fractional ghost number. Furthermore, what is called y there corresponds to x here.

JHEP10(2005)010

According to appendix B, the general solution to the first equation of (4.1) is given by

$$\Lambda_{\mu_1\ldots\mu_{s-1}}(x) = \sum_{m=0}^{s-1} A_{\mu_1\cdots\mu_{s-1}|\nu_1\cdots\nu_m} x^{\nu_1}\cdots x^{\nu_m}\,, \tag{4.2}$$

where the constant coefficients $A_{\mu_1\cdots\mu_{s-1}|\nu_1\cdots\nu_m}$ are required to have the symmetries of the Young tableaux

$$\begin{array}{l} \scriptstyle\mu \\ \scriptstyle\nu \end{array} \begin{array}{|c|c|c|c|c|c|}\hline \ & \cdots & \ & \ & \cdots & \ \\ \hline \ & \cdots & \ & \multicolumn{3}{|c}{} \\ \cline{1-3} \end{array} \quad . \tag{4.3}$$

The second equation of (4.1) then requires these tableaux to be traceless in the columns, while the last equation arises as a compatibility condition for the first two and requires the tableaux to be traceless in the second row. Finally, if the trace constraint on the gauge parameter is imposed, the tableaux are required to be traceless in the first row as well, and thus to be completely traceless.

4.2 Curved space

Let $\mathcal{M}$ be De Sitter or anti-De Sitter space, defined through the embedding $z^2 \equiv \eta_{\alpha\beta}z^\alpha z^\beta = \varepsilon L^2$ in flat $n+1$ dimensional space M^{n+1}, with $\varepsilon = 1$ for De Sitter and $\varepsilon = -1$ for anti-De Sitter space respectively. In describing tensor fields on $(A)dS$ we closely follow the approach of [17]. Take functions x^μ on M^{n+1} such that $z^\alpha \frac{\partial}{\partial z^\alpha} x^\mu = 0$ and $x^\mu, r = \sqrt{\varepsilon z^2}$ can be taken as local coordinates on M^{n+1} (minus the origin). It follows that, restricted to $\mathcal{M}$, the functions x^μ are local coordinates on $\mathcal{M}$. Conversely, any admissible local coordinates x'^μ on $\mathcal{M}$ can be lifted to such coordinate functions on M^{n+1}. These definitions imply that the invertible change of coordinates $z^\alpha = z^\alpha(r, x^\mu)$ satisfies

$$\begin{aligned} z_A\frac{\partial z^A}{\partial x^\mu} &= 0\,, & \frac{\partial x^\mu}{\partial z^A}\frac{\partial z^A}{\partial x^\nu} &= \delta^\mu_\nu\,, \\ \frac{\partial z^A}{\partial r} &= \tfrac{z^A}{r}\,, & \frac{\partial z^A}{\partial x^\mu}\frac{\partial x^\mu}{\partial z^B} + \varepsilon\frac{z^A z_B}{r^2} &= \delta^A_B\,. \end{aligned} \tag{4.4}$$

In addition to the embedding map $\mathcal{M} \to M^{n+1}$, one defines the projection $M^{n+1} \to \mathcal{M}$ which sends a point with coordinates r, x^μ to a point with coordinates x^μ, L. Using the projection one can establish a one-to-one correspondence between covariant tensor fields on $\mathcal{M}$ and their images on M^{n+1} under the pullback of the projection map. It can be useful to consider also the modified map given by

$$\Lambda_{A_1\ldots A_{s-1}}(z) = \left(\frac{\sqrt{\varepsilon z^2}}{L}\right)^N \frac{\partial x_L^{\mu_1}}{\partial z^{A_1}}\cdots\frac{\partial x_L^{\mu_{s-1}}}{\partial z^{A_{s-1}}}\Lambda_{\mu_1\ldots\mu_{s-1}}(x(z))\,. \tag{4.5}$$

where $\frac{\partial x^\mu_L}{\partial z^A}$ is $\frac{\partial x^\mu}{\partial z^A}$ evaluated at $r = L$. For $N = -s+1$ this coincides with the pullback of the projection map, which can be seen using

$$z^A\frac{\partial}{\partial z^A}\left(\frac{\partial x^\mu}{\partial z^B}\right) = r\frac{\partial}{\partial r}\left(\frac{\partial x^\mu}{\partial z^B}\right) = -\frac{\partial x^\mu}{\partial z^B} \tag{4.6}$$

JHEP10(2005)010

which in turn implies $\frac{\partial x^\mu}{\partial z^A} = (\frac{r}{L})^{-1} \frac{\partial x_L^\mu}{\partial z^A}$. The tensor $\Lambda_{A_1 \ldots A_{s-1}}(z)$ defined by (4.5) satisfies

$$z^{A_i} \Lambda_{A_1 \ldots A_i \ldots A_{s-1}}(z) = 0\,, \;\; z^B \frac{\partial}{\partial z^B} \Lambda_{A_1 \ldots A_i \ldots A_{s-1}}(z) = N \Lambda_{A_1 \ldots A_i \ldots A_{s-1}}(z)\,. \tag{4.7}$$

There is a one-to-one map from covariant tensor fields on $\mathcal{M}$ onto those on M^{n+1} satisfying (4.7). In terms of components the inverse map is given by

$$\Lambda_{\mu_1 \ldots \mu_{s-1}}(x) = \frac{\partial z_L^{A_1}}{\partial x_1^\mu} \ldots \frac{\partial z_L^{A_{s-1}}}{\partial x^{\mu_{s-1}}} \Lambda_{A_1 \ldots A_{s-1}}(z_L)\,. \tag{4.8}$$

where $z_L^A = z^A|_{r=L}$. Note also that $\frac{z^A}{r} = \frac{z_L^A}{L}$.

For $N = s-1$, the definition (4.5) implies

$$\partial_{(A_1} \Lambda_{A_2 \ldots A_s)}(z) = \left(\frac{\varepsilon z^2}{L^2} \right)^{s-1} \frac{\partial x^{\mu_1}}{\partial z^{A_1}} \ldots \frac{\partial x^{\mu_s}}{\partial z^{A_s}} \nabla_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)}(x)\,. \tag{4.9}$$

as can be seen by direct computation using the fact that the Levi-Civita connection on $\mathcal{M}$ is given in terms of the embedding by

$$\Gamma^\rho_{\mu\nu}(x) = \frac{\partial x_L^\rho}{\partial z^B} \frac{\partial^2 z_L^B}{\partial x^\mu \partial x^\nu} = \frac{\partial x^\rho}{\partial z^B} \frac{\partial^2 z^B}{\partial x^\mu \partial x^\nu}\,. \tag{4.10}$$

Hence,

$$\nabla_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)}(x) = 0 \tag{4.11}$$

implies

$$\partial_{(A_1} \Lambda_{A_2 \ldots A_s)}(z) = 0\,. \tag{4.12}$$

According to the previous subsection, the general solution to (4.12) is given by

$$\Lambda_{A_1 \ldots A_{s-1}}(z) = \sum_{m=0}^{s-1} A_{A_1 \ldots A_{s-1}|B_1 \ldots B_m} z^{B_1} \ldots z^{B_m}\,, \tag{4.13}$$

where the constant coefficients $A_{A_1 \ldots A_{s-1}|B_1 \ldots B_m}$ have again the symmetries of two row Young tableaux, the indexes running now over $n+1$ values. Only the last term with $m = s-1$ and involving, for $s > 1$, a Young tableau with two rows of equal length $s-1$,

$$\begin{array}{c} \scriptstyle A \\ \scriptstyle B \end{array} \begin{array}{|c|c|c|} \hline \; & \;\;\cdots\;\; & \; \\ \hline \; & \;\;\cdots\;\; & \; \\ \hline \end{array}\;\;, \tag{4.14}$$

satisfies the conditions (4.7) with $N = s-1$. The solutions to $\nabla_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)}(x) = 0$ are thus explicitly given by

$$\Lambda_{\mu_1 \ldots \mu_{s-1}}(x) = \frac{\partial z_L^{A_1}}{\partial x_1^\mu} \ldots \frac{\partial z_L^{A_{s-1}}}{\partial x^{\mu_{s-1}}} A_{A_1 \ldots A_{s-1}|B_1 \ldots B_{s-1}} z_L^{B_1} \ldots z_L^{B_{s-1}}\,. \tag{4.15}$$

and are characterized by a *single* 2 row rectangular Young tableau of length $s-1$ in $n+1$ dimensions.

Using

$$\frac{\partial^2 x^\mu}{\partial z^A \partial z^B} = -\frac{\partial x^\mu}{\partial z^A}\frac{\varepsilon z_B}{r^2} - \frac{\partial x^\mu}{\partial z^B}\frac{\varepsilon z_A}{r^2} - \frac{\partial x^\mu}{\partial z^C}\frac{\partial x^\nu}{\partial z^A}\frac{\partial x^\lambda}{\partial z^B}\frac{\partial^2 z^C}{\partial x^\nu \partial x^\lambda}\,, \tag{4.16}$$

definition (4.5) for $N = s-1$ implies

$$\begin{aligned}\partial^A \Lambda_{AA_2\ldots A_{s-1}} = \left(\frac{\varepsilon z^2}{L^2}\right)^{s-2} \Bigg[&\frac{\partial x^{\mu_2}}{\partial z^{A_2}}\cdots\frac{\partial x^{\mu_{s-1}}}{\partial z^{A_{s-1}}}\nabla^\mu \Lambda_{\mu\mu_2\ldots\mu_{s-1}} - \\ &-\left(\frac{\varepsilon z_{A_2}}{r^2}\frac{\partial x^{\mu_3}}{\partial z^{A_3}}\cdots\frac{\partial x^{\mu_{s-1}}}{\partial z^{A_{s-1}}} + \frac{\partial x^{\mu_3}}{\partial z^{A_2}}\frac{\varepsilon z_{A_3}}{r^2}\frac{\partial x^{\mu_4}}{\partial z^{A_4}}\cdots\frac{\partial x^{\mu_{s-1}}}{\partial z^{A_{s-1}}} + \cdots + \right.\\ &\left. + \frac{\partial x^{\mu_3}}{\partial z^{A_2}}\cdots\frac{\partial x^{\mu_{s-1}}}{\partial z^{A_{s-2}}}\frac{\varepsilon z_{A_{s-1}}}{r^2}\right)\Lambda'_{\mu_3\ldots\mu_{s-1}}\Bigg]\,. \end{aligned} \tag{4.17}$$

By contracting with $\eta^{A_1A_p}$ and using $g^{\mu_1\mu_2} = \frac{\partial x_L^{\mu_1}}{\partial z^{A_1}}\eta^{A_1A_2}\frac{\partial x_L^{\mu_2}}{\partial z^{A_2}}$, definition (4.5) for $N = s-1$ also implies

$$\Lambda^A_{A_2\ldots A_{p-1}AA_{p+1}\ldots A_{s-1}} = \left(\frac{\varepsilon z^2}{L^2}\right)^{s-2}\frac{\partial x^{\mu_3}}{\partial z^{A_2}}\cdots\frac{\partial x^{\mu_p}}{\partial z^{A_{p-1}}}\frac{\partial x^{\mu_{p+1}}}{\partial z^{A_{p+1}}}\cdots\frac{\partial x^{\mu_{s-1}}}{\partial z^{A_{s-1}}}\Lambda'_{\mu_3\ldots\mu_{s-1}}\,. \tag{4.18}$$

It follows that the equations

$$\nabla^\mu \Lambda_{\mu\mu_2\ldots\mu_{s-1}} = 0 \tag{4.19}$$

are equivalent to

$$z^2\partial^A\Lambda_{AA_2\ldots A_{s-1}} + \sum_{p=2}^{s-1} z_{A_p}\Lambda^A_{A_2\ldots A_{p-1}AA_{p+1}\ldots A_{s-1}} = 0\,. \tag{4.20}$$

This equation imposes the following constraint on the rectangular 2 row Young tableau:

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}(s)}\Bigg[(s-1)\eta_{B_{\sigma(1)}B_{\sigma(2)}}A^A_{A_2\ldots A_{s-1}|AB_{\sigma(3)}\ldots B_{\sigma(s)}} + \\ &+\sum_{p=2}^{s-1}\eta_{A_pB_{\sigma(1)}}A'_{A_2\ldots A_{p-1}A_{p+1}\ldots A_{s-1}|B_{\sigma(2)}B_{\sigma(3)}\ldots B_{\sigma(s)}}\Bigg] = 0\,.\end{aligned} \tag{4.21}$$

The equation

$$\left[\Box + \frac{(s-1)(3-s-n)}{L^2}\right]\Lambda_{\mu_1\mu_2\ldots\mu_{s-1}} + \frac{(s-1)(s-2)}{L^2}g_{(\mu_1\mu_2}\Lambda'_{\mu_3\ldots\mu_{s-1})} = 0\,, \tag{4.22}$$

is then automatically satisfied since it is the compatibility condition for (4.11) and (4.19).

If the trace condition is imposed to start with, the problem simplifies since the rectangular 2 row Young tableau is required to be traceless in the first row from the start. Equations (4.20) then simply require this Young tableau to be traceless in the columns as well, while the compatibility condition reduces, as in the flat case, to requiring the tableau to be traceless in the second row, and thus to be completely traceless.

5. Surface charges for higher spin gauge fields

The Euler-Lagrange derivatives associated to action (2.11) are explicitly given by

$$
\begin{aligned}
\mathcal{L}_\varphi^{\mu_1\dots\mu_s} &= \frac{\sqrt{|g|}}{s!}\Bigg[\Box\varphi^{\mu_1\dots\mu_s} - s\nabla^{(\mu_1}C^{\mu_2\dots\mu_s)} - \frac{1}{L^2}\times \\
&\qquad\times\Bigg[4s(s-1)g^{(\mu_1\mu_2}D^{\mu_3\dots\mu_s)} - \qquad (5.1)\\
&\qquad\quad - s(s-1)g^{(\mu_1\mu_2}\varphi'^{\mu_3\dots\mu_s)} + [(2-s)(3-n-s)-s]\varphi^{\mu_1\dots\mu_s}\Bigg]\Bigg],\\
\mathcal{L}_C^{\mu_1\dots\mu_{s-1}} &= \frac{\sqrt{|g|}}{(s-1)!}\left[-C^{\mu_1\dots\mu_{s-1}} + \nabla_\mu\varphi^{\mu\mu_1\dots\mu_{s-1}} - (s-1)\nabla^{(\mu_1}D^{\mu_2\dots\mu_{s-1})}\right], \qquad (5.2)\\
\mathcal{L}_D^{\mu_1\dots\mu_{s-2}} &= \frac{\sqrt{|g|}}{(s-2)!}\Bigg[-\Box D^{\mu_1\dots\mu_{s-2}} + \nabla_\mu C^{\mu\mu_1\dots\mu_{s-1}} - \frac{1}{L^2}\times\\
&\qquad\times\Bigg[4\varphi'^{\mu_1\dots\mu_{s-2}} + (s-2)(s-3)g^{(\mu_1\mu_2}D'^{\mu_3\dots\mu_{s-2})} - \\
&\qquad\quad - [s(n+s-2)+6]D^{\mu_1\dots\mu_{s-2}}\Bigg]\Bigg].
\end{aligned}
$$

Using the Noether identities

$$
\begin{aligned}
-s\nabla_\mu\mathcal{L}_\varphi^{\mu\mu_1\dots\mu_{s-1}} - \nabla^{(\mu_1}\mathcal{L}_D^{\mu_2\dots\mu_{s-1})} + \left[\Box + \frac{(s-1)(3-s-n)}{L^2}\right]\mathcal{L}_C^{\mu_1\dots\mu_{s-1}} + &\\
+\frac{(s-1)(s-2)}{L^2}g^{(\mu_1\mu_2}\mathcal{L}_C'^{\mu_3\dots\mu_{s-1})} &= 0\,, \qquad (5.3)
\end{aligned}
$$

the weakly vanishing Noether current j^β_Λ defined by

$$
\begin{aligned}
&s\mathcal{L}_\varphi^{\mu_1\dots\mu_s}\nabla_{(\mu_1}\Lambda_{\mu_2\dots\mu_s)} + \mathcal{L}_D^{\mu_1\dots\mu_{s-2}}\nabla^\mu\Lambda_{\mu\mu_1\dots\mu_{s-2}} + \\
&+\mathcal{L}_C^{\mu_1\dots\mu_{s-1}}\left[\left[\Box + \frac{(s-1)(3-s-n)}{L^2}\right]\Lambda_{\mu_1\dots\mu_{s-1}} + \right.\\
&\qquad\left. + \frac{(s-1)(s-2)}{L^2}g_{(\mu_1\mu_2}\Lambda'_{\mu_3\dots\mu_{s-1})}\right] = \partial_\beta j^\beta_\Lambda\,, \qquad (5.4)
\end{aligned}
$$

is found to be

$$
\begin{aligned}
j^\beta_\Lambda = {}& s\mathcal{L}_\varphi^{\beta\mu_1\dots\mu_{s-1}}\Lambda_{\mu_1\dots\mu_{s-1}} + \mathcal{L}_D^{\mu_2\dots\mu_{s-1}}\Lambda^\beta_{\ \mu_2\dots\mu_{s-1}} + \\
&+\mathcal{L}_C^{\mu_1\dots\mu_{s-1}}\nabla^\beta\Lambda_{\mu_1\dots\mu_{s-1}} - \nabla^\beta\mathcal{L}_C^{\mu_1\dots\mu_{s-1}}\Lambda_{\mu_1\dots\mu_{s-1}}\,. \qquad (5.5)
\end{aligned}
$$

Applying previous results derived for generic gauge theories [33], non trivial on-shell conserved $n-2$-forms are associated with non trivial reducibility parameters and thus with divergence free Killing tensors $\Lambda_{\mu_1\dots\mu_{s-1}}(x)$. Explicitly, the $n-2$ forms $k_\Lambda = k_\Lambda^{[\alpha\beta]}(d^{n-2}x)_{\alpha\beta}$, with $(d^{n-2}x)_{\alpha\beta} = \frac{1}{2(n-2)!}\epsilon_{\alpha\beta\mu_3\dots\mu_n}dx^{\mu_3}\dots dx^{\mu_n}$, $\epsilon_{01\dots n-1}=1$, satisfy

$$
\partial_\alpha k_\Lambda^{[\alpha\beta]} = j^\beta_\Lambda \qquad (5.6)
$$

JHEP10(2005)010

and are constructed out of j^β_Λ according to

$$k^{[\alpha\beta]}_\Lambda = \frac{1}{2}\phi^i \frac{\partial j^\beta_\Lambda}{\partial \phi^i_\alpha} + \left(\frac{2}{3}\phi^i_\lambda - \frac{1}{3}\phi^i\partial_\lambda\right)\frac{\partial^S j^\beta_\Lambda}{\partial \phi^i_{\lambda\alpha}} - (\alpha \longleftrightarrow \beta)\,, \tag{5.7}$$

where $\phi^i \equiv (\varphi_{\mu_1\dots\mu_s}, C_{\mu_1\dots\mu_{s-1}}, D_{\mu_1\dots\mu_{s-2}})$, the subscripts on ϕ^i denoting derivatives and $\partial^S\phi^j_{\alpha\gamma}/\partial\phi^i_{\delta\beta} = \delta^j_i\delta^\delta_{(\alpha}\delta^\beta_{\gamma)}$. Direct computation gives

$$\begin{aligned} k^{[\alpha\beta]}_\Lambda = \frac{\sqrt{|g|}}{(s-1)!}\Big[&\nabla^\alpha\varphi^{\beta\mu_1\dots\mu_{s-1}}\Lambda_{\mu_1\dots\mu_{s-1}} + (s-1)\nabla^\beta D^{\mu_1\dots\mu_{s-2}}\Lambda^\alpha_{\mu_1\dots\mu_{s-2}} + \\ &+\varphi^{\alpha\mu_1\dots\mu_{s-1}}\nabla^\beta\Lambda_{\mu_1\dots\mu_{s-1}} + (s-1)D^{\mu_1\dots\mu_{s-2}}\nabla^\alpha\Lambda^\beta_{\mu_1\dots\mu_{s-2}} + \\ &+(s-1)C^{\alpha\mu_1\dots\mu_{s-2}}\Lambda^\beta_{\mu_1\dots\mu_{s-2}} - (\alpha \longleftrightarrow \beta)\Big]\,. \end{aligned} \tag{5.8}$$

Comments.

(i) The fields $C_{\mu_1\dots\mu_{s-1}}$ are auxiliary since they can be eliminated by their own equations of motion, $\mathcal{L}^{\mu_1\dots\mu_{s-1}}_C = 0$. If this is done on the level of j^β_Λ before the computation of $k^{[\alpha\beta]}_\Lambda$, both expressions coincide only after using that $\Lambda_{\mu_1\dots\mu_{s-1}}(x)$ are divergence free Killing tensors.

(ii) The conserved $n-2$ forms of the Fronsdal theory for higher spin gauge fields are obtained by substituting $D = \frac{1}{2}\varphi'$ in k_Λ, with φ taken to be double traceless and Λ traceless.

(iii) After these steps, the $n-2$ forms k_Λ for the spin 1 and 2 cases can easily be seen to coincide with the Abbott and Deser expressions in the form given in [33, section 6].

6. Conclusion and perspectives

In this paper we have computed the non trivial surface charges for higher spin gauge fields in constant curvature spacetimes by relating them to divergence free dynamical Killing tensors. In the anti-De Sitter case where definite progress towards constructing non trivial interactions has been made (see e.g. [31] for a review), it remains to be seen if the surface charges for higher spin fields can play as prominent a role as the Abbott-Deser charges do in the case of asymptotically anti-De Sitter gravity.

It is quite intriguing to realize that the computation of a physically relevant cohomology group in the field theory effectively reduces to the computation of a cohomology group of the first quantized model: in this article, we have shown in particular that the surface charges, or more precisely, that the characteristic cohomology group $H^{n-2}_{\text{weak}}(d)$ in the field theory is isomorphic to the BRST cohomology group $H^{-3/2}(\Omega)$ of the first quantized particle model underlying the theory of massless higher spin gauge fields. In other words, for the cohomology groups $H^{-3/2}(\Omega)$ for which no clear physical interpretation seems to have been known (see e.g. of [52, chapter 11.1.2]), we have been able to find such an interpretation in the associated field theory.

JHEP10(2005)010

A reason why the cohomology group $H^{-3/2}(\Omega)$ has not attracted more interest so far is that this cohomology is normally argued to vanish [23]. As usual in such cases, this is because the cohomology of Ω is computed in different spaces: in [23], the cohomology is computed in momentum space at non zero momentum, whereas in [28], it is computed in x-space and is described by polynomials in x, which means that it is concentrated precisely at zero momentum.[4]

A similar analysis can of course be repeated in the BRST formulation of string theory. In this way, one should be able to construct meaningful surfaces charges in string field theory associated with negative ghost number cohomology classes of the BRST charge in the zero momentum sector. We plan to analyze this question in more details in the near future.

Acknowledgments

Useful discussions with X. Bekaert, G. Bonelli, G. Compère, A. Semikhatov and I. Tipunin are gratefully acknowledged. The work of G.B. and N.B. is supported in part by a "Pôle d'Attraction Interuniversitaire" (Belgium), by IISN-Belgium, convention 4.4505.86, by Proyectos FONDECYT 1970151 and 7960001 (Chile) and by the European Commission program MRTN-CT-2004-005104, in which these authors are associated to V.U. Brussels. The work of M.G. is supported by RFBR grant 04-01-00303 and by the grant LSS-1578.2003.2.

Note added: while completing this work, reference [53] appeared where, in particular, the dynamical Killing tensor equations in flat spacetimes have been solved under simplifying assumptions. We also understand that the standard Killing tensor equations in constant curvature spaces have been solved previously in [54] and that the connection with rectangular 2 row Young tableaux of length $s-1$ in $n+1$ dimensions, both in flat and in constant curvature spaces, has been made previously in [55].

A. Cauchy order of higher spin gauge fields

Because of gauge invariance, the (left hand sides of the) equations of motion are not all independent functions on the space where the fields and their derivatives are considered as independent coordinates: they satisfy the so-called Noether identities. As a consequence, we can split the equations of motion into two groups, the "independent equations" L_a and "the dependent equations" L_Δ (the dependent equations hold as a consequences of the independent ones: $L_a = 0$ implies $L_\Delta = 0$). We will show below that the fields and their derivatives can be split into two groups, "independent coordinates" y_A, which are not constrained by the equations of motions, and "dependent coordinates" z_a, which are such that $L_a = 0$ can be solved for z_a in terms of the y_A. This implies in particular that the gauge transformations (3.2) provide a generating set of irreducible gauge transformations.

[4] G.B. is grateful to G. Bonelli for a useful discussion on this point.

	$\mathcal{L}_\Delta$	$\mathcal{L}_a$	z_a	y_A
V^0	$\emptyset$	$\emptyset$	$\emptyset$	$\varphi_{(\mu)}, C_{(\nu)}, D_{(\lambda)}$
V^1	$\emptyset$	$\mathcal{L}_C^{(\nu)}$	$\varphi_{0(\nu)\|0}$	$\varphi_{(k)\|0}, \varphi_{(\mu)\|i}, C_{(\nu)\|\alpha}$, $D_{(\lambda)\|\alpha}$
V^2	$\emptyset$	$\mathcal{L}_\varphi^{(\mu)}, \mathcal{L}_D^{(\lambda)}, \partial_\beta \mathcal{L}_C^{(\nu)}$	$\varphi_{0(\nu)\|0\beta}, \varphi_{(k)\|00}, \varphi_{0(\nu)\|11}$, $D_{(\lambda)\|00}$	$\varphi_{(k)\|m\rho}, \varphi_{0(\nu)\|\bar{l}p}, C_{(\nu)\|\alpha_1\alpha_2}$, $D_{(\lambda)\|m\rho}$
V^3	$\partial_0 \mathcal{L}_\varphi^{0(\nu)}$	$\partial_{\beta_1\beta_2} \mathcal{L}_C^{(\nu)}, \partial_\gamma \mathcal{L}_\varphi^{(k)}$, $\partial_m \mathcal{L}_\varphi^{0(\nu)}, \partial_\gamma \mathcal{L}_D^{(\lambda)}$	$\varphi_{0(\nu)\|0\beta_1\beta_2}, \varphi_{(k)\|00\gamma}, \varphi_{0(\nu)\|11n}$, $D_{(\lambda)\|00\gamma}$	$\varphi_{(k)\|m_1m_2\rho}, \varphi_{0(\nu)\|\bar{l_1}\bar{l_2}p}, C_{(\nu)\|\alpha_1\alpha_2\alpha_3}$, $D_{(\lambda)\|m_1m_2\rho}$
. . .				
V^k	$\partial_{(\delta)0} \mathcal{L}^{0(\nu)}$	$\partial_{(\beta)} \mathcal{L}_C^{(\nu)}, \partial_{(\gamma)} \mathcal{L}_\varphi^{(k)}$, $\partial_{(n)} \mathcal{L}_\varphi^{0(\nu)}, \partial_{(\gamma)} \mathcal{L}_D^{(\lambda)}$	$\varphi_{0(\nu)\|0(\beta)}, \varphi_{(k)\|00(\gamma)}$ $\varphi_{0(\nu)\|11(n)}, D_{(\lambda)\|00(\gamma)}$	$\varphi_{(k)\|(m)\rho}, \varphi_{0(\nu)\|(\bar{l})p}$, $C_{(\nu)\|(\alpha)}, D_{(\lambda)\|(m)\rho}$

Table 1: Split of variables

Without loss of generality we can focus in the following on flat space. Indeed, the splits of equations and variables in flat space will also be valid in spaces with non vanishing constant curvature because the terms with the highest number of derivatives in the Noether identities and in the equations of motions in constant curvature spaces are the same in the two cases. We will start by discussing the model without trace constraint.

The equations of motion in flat space are explicitly given by

$$
\begin{aligned}
\Box \varphi_{\mu_1 \ldots \mu_s} &= s \partial_{(\mu_1} C_{\mu_2 \ldots \mu_s)} \,, \\
\partial^\mu \varphi_{\mu\mu_2 \ldots \mu_s} - (s-1) \partial_{(\mu_2} D_{\mu_3 \ldots \mu_s)} &= C_{\mu_2 \ldots \mu_s} \,, \\
\Box D_{\mu_1 \ldots \mu_{s-2}} &= \partial^\mu C_{\mu\mu_1 \ldots \mu_{s-2}} \,,
\end{aligned}
\tag{A.1}
$$

and the corresponding gauge transformations by

$$
\begin{aligned}
\delta_\Lambda \varphi_{\mu_1 \ldots \mu_s} &= s \partial_{(\mu_1} \Lambda_{\mu_2 \ldots \mu_s)} \,, \\
\delta_\Lambda D_{\mu_1 \ldots \mu_{s-2}} &= \partial^\mu \Lambda_{\mu\mu_1 \ldots \mu_{s-2}} \,, \\
\delta_\Lambda C_{\mu_1 \ldots \mu_{s-1}} &= \Box \Lambda_{\mu_1 \ldots \mu_{s-1}} \,.
\end{aligned}
\tag{A.2}
$$

In terms of the Euler-Lagrange derivatives $\mathcal{L}_\varphi^{\mu_1 \ldots \mu_s}$, $\mathcal{L}_C^{\mu_1 \ldots \mu_{s-1}}$ $\mathcal{L}_D^{\mu_1 \ldots \mu_{s-2}}$ associated to action (2.11), the Noether identities are given by

$$
s\, \partial_\mu \mathcal{L}_\varphi^{\mu\mu_2 \ldots \mu_s} - \Box \mathcal{L}_C^{\mu_1 \ldots \mu_{s-1}} + \partial^{(\mu_2} \mathcal{L}_D^{\mu_3 \ldots \mu_s)} = 0 \,. \tag{A.3}
$$

These identities can be solved for $\partial_0 \mathcal{L}_\varphi^{0\mu_2 \ldots \mu_s}$ and their derivatives, which constitute the L_Δ. The remaining equations and their derivatives are the L_a, they are all independent because they can be solved for the z_a listed in table 1. The remaining fields and their derivatives are the y_A. We have separated the tensor indexes on the fields from the indexes indicating derivatives by a vertical bar. The Greek indexes go from 0 to $n-1$, the Latin indexes from 1 to $n-1$, while the barred Latin indexes go from 2 to $n-1$. The multi-indexes $(\mu), (k), (\nu), (\lambda)$ are of orders $|\mu| = s = (k), |\nu| = s-1, |\lambda| = s-2$, while the indexes $(\alpha), (\beta), (m), (\bar{l}), (\gamma), (n), (\delta)$ are of orders $|\alpha| = k, |\beta| = k-1 = |m| = |\bar{l}|, |\gamma| = k-2 = |n|, |\delta| = k-3$. The listed y_A have the property we wanted to show, namely $\partial_{\bar{l}} y_A \subset y_A$.

In the case where the fields satisfy trace constraints, we have simply to require the C and D fields and all their derivatives in table 1 to be traceless fields, while the condition $\varphi' = 2D$ is implemented by removing the components φ_{11} and all their derivatives from the y_A. The Cauchy order is unchanged.

B. Killing tensors in flat space

Consider real functions $\Lambda(a^{\dagger\mu}, x^\mu)$, where the dependence of Λ on the variables $a^{\dagger\mu}$ is assumed to be polynomial, while the dependence on x^μ is assumed to be smooth. Let us introduce the operators

$$S^\dagger = a^{\dagger\mu}\frac{\partial}{\partial x^\mu}\,, \quad N_a = a^{\dagger\mu}\frac{\partial}{\partial a^{\dagger\mu}}\,, \quad \Box = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\,, \quad S = \eta^{\mu\nu}\frac{\partial}{\partial a^{\dagger\mu}}\frac{\partial}{\partial x^\nu}\,. \tag{B.1}$$

The gauge parameters for the models of level $s-1$ can then be identified with functions Λ_{s-1} satisfying $N_a\Lambda_{s-1} = (s-1)\Lambda_{s-1}$ and equations (4.1) are equivalent to

$$S^\dagger\Lambda_{s-1} = 0\,, \qquad S\Lambda_{s-1} = 0\,, \qquad \Box\Lambda_{s-1} = 0\,. \tag{B.2}$$

Introducing the additional operators

$$\bar{S}^\dagger = x^\mu\frac{\partial}{\partial a^{\dagger\mu}}\,, N_x = x^\mu\frac{\partial}{\partial x^\mu}\,, \qquad H = N_a - N_x\,, \tag{B.3}$$

the subset $S^\dagger, \bar{S}^\dagger, H$ form an *sl(2)* algebra,

$$[S^\dagger, \bar{S}^\dagger] = H\,, \quad [h, S^\dagger] = 2S^\dagger\,, \quad [H, \bar{S}^\dagger] = -2\bar{S}^\dagger\,. \tag{B.4}$$

represented on polynomials in $a^\dagger$ with coefficients in smooth functions of x with $S^\dagger$ and $\bar{S}^\dagger$ acting as creation and destruction operators respectively. The results now follow from standard arguments used to describe $sl(2)$ representations which we spell out explicitly in order to be self-contained.

Because they involve exactly $s-1$ variables $a^{\dagger\mu}$, the parameters Λ_{s-1} also satisfy the condition

$$(\bar{S}^\dagger)^k\Lambda_{s-1} = 0\,, \qquad \forall k \geqslant s\,. \tag{B.5}$$

The solutions Λ_{s-1} to $S^\dagger\Lambda_{s-1} = 0$ can then be classified according to the lowest integer $0 \leqslant m \leqslant s-1$ such that $(\bar{S}^\dagger)^{s-m}\Lambda_{s-1} = 0$. Applying $S^\dagger$ to the latter equation, one can easily show using the commutation relations that

$$\left[(s-m)(s-m-1)(\bar{S}^\dagger)^{s-m-1} + (s-m)H(\bar{S}^\dagger)^{s-m-1}\right]\Lambda_{s-1} = 0\,. \tag{B.6}$$

Using the definition of H, this relation gives

$$N_x\left((\bar{S}^\dagger)^{s-m-1}\Lambda_{s-1}\right) = (s-1)(\bar{S}^\dagger)^{s-m-1}\Lambda_{s-1}\,, \tag{B.7}$$

and thus $N_x\Lambda_{s-1} = m\Lambda_{s-1}$. The gauge parameters of the different classes of solutions characterized by the integer $0 \leqslant m \leqslant s-1$ are thus of homogeneity m in x^μ. Injecting $\Lambda_{s-1}(x, a^\dagger) = a^{\dagger\mu_1}\ldots a^{\dagger\mu_{s-1}}A_{\mu_1\ldots\mu_{s-1}|\nu_1\ldots\nu_m}x^{\nu_1}\ldots x^{\nu_m}$ in the equation $S^\dagger\Lambda_{s-1} = 0$ then

JHEP10(2005)010

implies that the constants $A_{\mu_1\ldots\mu_{s-1}|\nu_1\ldots\nu_m}\Lambda_{s-1}$ have the symmetries of the Young tableau

$$\begin{array}{c|c|c|c|c|c|c|}
\cline{2-7}
\mu & & \cdots & & & \cdots & \\
\cline{2-7}
\nu & & \cdots & & \multicolumn{3}{c}{} \\
\cline{2-4}
\end{array} \tag{B.8}$$

where the first row is of length $s-1$ and the second row is of length $0 \leqslant m \leqslant s-1$.

That $S\Lambda_{s-1} = 0$ requires the tableaux to be traceless in the rows is now obvious, while the commutation relation $[S, S^\dagger] = -\Box$ implies that $\Box\Lambda_{s-1} = 0$ is a consistency condition for $S^\dagger\Lambda_{s-1} = 0 = S\Lambda_{s-1} = 0$.

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JHEP10(2005)010